ous combinations of a_1, σ_k and a_2, allowing a_1 to vary from 0.2 to 0.3, σ_k from 0.4 to 0.7 and a_2 from 1.0 to 2.4. The results indicated that the centerline mean velocity changed by less than 10% and the maximum shear changed by approximately 30%.

Conclusion

The use of the turbulent kinetic energy equation reduces the uncertainty introduced by the phenomenological model of eddy viscosity. The coefficients of $a_1 = 0.3$, $\sigma_k = 0.7$ and $a_2 = 1.5$ provide reasonable predictions for most of the studied engineering problems. However, detailed turbulence measurements are very much needed if the turbulence energy approach can be of any significance in contributing to the basic understanding of the turbulence phenomenon.

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Relative Motion of Two Particles in Elliptic Orbits

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Nomenclature

$\mathbf{r}$ = position vector at time t, $r = |\mathbf{r}|$
$\mathbf{v}$ = velocity vector at time t, $v = |\mathbf{v}|$
E = eccentric anomaly at time t
a = semimajor axis, $b = 1/a$, $c = a^{1/2}$
k^2 = μ = gravitational constant
$\mathbf{r}\cdot\mathbf{v}$ = scalar product of $\mathbf{r}$ and $\mathbf{v}$

Over the past decade a number of papers[1–6] have presented approximate solutions to the problem of the relative motion of two particles in elliptic orbits in an inverse-square central force field. These papers have assumed that the relative position and velocity vectors are quite small compared to the position and velocity vectors of the particles, and several have further assumed one of the particles to be in a circular[1,2] or nearly circular[3,4] orbit. We derive below an exact solution to this problem, subject to no restrictions. The results have applications to problems of rendezvous, nonlinear error analysis, station keeping, targeting, surveillance, and satellite clustering.

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Solution

Kepler's equation for elliptic motion and the updating equations for position and velocity can be written in the form[7]

$$T = C + A(1 - \cos C) - B \sin C \tag{1}$$

$$\mathbf{r} = [1 - H(1 - \cos C)]\mathbf{r}_0 + [D(1 - \cos C) + N \sin C]\mathbf{v}_0 \tag{2}$$

$$\mathbf{v} = -(P \sin C)\mathbf{r}_0 + [1 - S(1 - \cos C)]\mathbf{v}_0 \tag{3}$$

where

$$A = \mathbf{r}_0\cdot\mathbf{v}_0/(kc),\ B = 1 - r_0 b,\ T = ktb/c$$

$$H = a/r_0,\ D = a(\mathbf{r}_0\cdot\mathbf{v}_0)/\mu,\ N = r_0 c/k$$

$$P = kc/(rr_0),\ S = a/r,\ C = E - E_0$$

and a zero subscript indicates the value at time 0.

Let subscript 1 on a symbol designate the value of that symbol for particle 1 in orbit 1 and subscript 2 the value for particle 2 in orbit 2. We define

$$\gamma = C_2 - C_1,\ \boldsymbol{\varepsilon} = \mathbf{r}_2 - \mathbf{r}_1,\ \boldsymbol{\lambda} = \mathbf{v}_2 - \mathbf{v}_1,\ \tau = T_2 - T_1$$

$$\alpha = A_2 - A_1$$

$$\beta = B_2 - B_1,\ \eta = H_2 - H_1,\ \delta = D_2 - D_1,\ \nu = N_3 - N_1$$

$$\rho = P_2 - P_1,\ \sigma = S_2 - S_1$$

If we place a subscript 1 on all symbols in Eqs. (1–3) and subtract the resulting equations from the set with subscripts 2 on all symbols, we obtain

$$T' = \gamma + A'(1 - \cos\gamma) - B' \sin\gamma \tag{4}$$

$$\boldsymbol{\varepsilon} = (1 - H_1F)\boldsymbol{\varepsilon}_0 + (D_1F + N_1G)\boldsymbol{\lambda}_0 - (H_2Q + \eta F)\mathbf{r}_{20} + (D_2Q + \delta F + N_2R + \nu G)\mathbf{v}_{20} \tag{5}$$

$$\boldsymbol{\lambda} = -P_1G\boldsymbol{\epsilon}_0 + (1 - S_1F)\boldsymbol{\lambda}_0 - (P_2R + \rho G)\mathbf{r}_{20} - (S_2Q + \sigma F)\mathbf{v}_{20} \tag{6}$$

where we have defined

$$F = 1 - \cos C_1,\ G = \sin C_1 \tag{7}$$

$$T' = \tau + \beta G - \alpha F \tag{8}$$

$$A' = A_2 \cos C_1 + B_2 \sin C_1 \tag{9}$$

$$B' = B_2 \cos C_1 - A_2 \sin C_1 \tag{10}$$

$$Q = \cos C_1(1 - \cos\gamma) + \sin C_1 \sin\gamma \tag{11}$$

$$R = \cos C_1 \sin\gamma - \sin C_1(1 - \cos\gamma) \tag{12}$$

To obtain equations for α, β, τ, η, δ, ν, ρ, and σ which do not suffer a loss of significant digits due to the subtraction of nearly equal numbers, we proceed as follows:

$$kcA = \mathbf{r}_0\cdot\mathbf{v}_0$$

$$k(c_2A_2 - c_1A_1) = \mathbf{r}_{20}\cdot\mathbf{v}_{20} - \mathbf{r}_{10}\cdot\mathbf{v}_{10}$$

$$k[c_2(A_2 - A_1) + A_1(c_2 - c_1)] = \mathbf{r}_{20}\cdot\mathbf{v}_{20} - \mathbf{r}_{10}\cdot\mathbf{v}_{10}$$

The last equation can be solved for α. Equations for the other quantities can be obtained in a similar manner. The full set follows:

$$kc_2\alpha = \mathbf{r}_{20}\cdot\mathbf{v}_{20} - \mathbf{r}_{10}\cdot\mathbf{v}_{10} - kA_1(c_2 - c_1) \tag{13}$$

$$\beta = r_{10}(b_1 - b_2) - b_2(r_{20} - r_{10}) \tag{14}$$

$$c_2\tau = -kt(b_1 - b_2) - T_1(c_2 - c_1) \tag{15}$$

$$r_{10}\eta = a_2 - a_1 - H_2(r_{20} - r_{10}) \tag{16}$$

$$\mu\delta = a_2(\mathbf{r}_{20}\cdot\mathbf{v}_{20} - \mathbf{r}_{10}\cdot\mathbf{v}_{10}) + (a_2 - a_1)\mathbf{r}_{10}\cdot\mathbf{v}_{10} \tag{17}$$

$$k\nu = c_2(r_{20} - r_{10}) + r_{10}(c_2 - c_1) \tag{18}$$

$$r_{10}r_1\rho = k(c_2 - c_1) - P_2[r_2(r_{20} - r_{10}) + r_{10}(r_2 - r_1)] \tag{19}$$

$$r_1\sigma = a_2 - a_1 - S_2(r_2 - r_1) \tag{20}$$

$$\mathbf{r}_{20}\cdot\mathbf{v}_{20} - \mathbf{r}_{10}\cdot\mathbf{v}_{10} = \boldsymbol{\lambda}_0\cdot\mathbf{r}_{10} + \boldsymbol{\epsilon}_0\cdot\mathbf{v}_{20} \tag{21}$$

$$r_{20} - r_{10} = \boldsymbol{\varepsilon}_0\cdot(\mathbf{r}_{10} + \mathbf{r}_{20})/(r_{10} + r_{20}) \tag{22}$$

$$r_2 - r_1 = \boldsymbol{\varepsilon}\cdot(\mathbf{r}_1 + \mathbf{r}_2)/(r_1 + r_2) \tag{23}$$

$$c_2 - c_1 = (a_2 - a_1)/(c_1 + c_2) \tag{24}$$

$$a_2 - a_1 = a_1a_2(b_1 - b_2) \tag{25}$$

$$b_1 - b_2 = 2(r_{20} - r_{10})/r_{10}r_{20} + \boldsymbol{\lambda}_0\cdot(\mathbf{v}_{10} + \mathbf{v}_{20})/\mu \tag{26}$$

In the derivation of Eq. (26) use was made of

$$b = 2/r - v^2/\mu \tag{27}$$

Summary

Given $\mathbf{r}_{10}$, $\mathbf{v}_{10}$, $\boldsymbol{\varepsilon}_0$, $\boldsymbol{\lambda}_0$ at time 0, the steps for finding $\boldsymbol{\varepsilon}$ and $\boldsymbol{\lambda}$ at time t follow: 1) compute $\mathbf{r}_{20} = \mathbf{r}_{10} + \boldsymbol{\varepsilon}_0$, $\mathbf{v}_{20} = \mathbf{v}_{10} + \boldsymbol{\lambda}_0$, $r_{10} = (\mathbf{r}_{10}\cdot\mathbf{r}_{10})^{1/2}$, $v_{10}^2 = \mathbf{v}_{10}\cdot\mathbf{v}_{10}$; 2) compute A_1, B_1, T_1, H_1, D_1, N_1 [equation after (3)], a_1 and b_1 from Eq. (27); 3) solve Kepler's Eq. (1) for C_1; if C_1 is given rather than t, Eq. (1) is used to compute T_1; 4) compute $\mathbf{r}_1$ from Eq. (2) and $r_1 = (\mathbf{r}_1\cdot\mathbf{r}_1)^{1/2}$; 5) compute P_1, S_1, A_2, B_2, H_2, D_2, N_2 [equations after (3)], a_2 and b_2 from Eq. (27); 6) compute in order $\mathbf{r}_{20}\cdot\mathbf{v}_{20} - \mathbf{r}_{10}\cdot\mathbf{v}_{10}$, $r_{20} - r_{10}$, $b_1 - b_2$, $a_2 - a_1$, $c_2 - c_1$ from Eqs. (21, 22, 26, 25, and 24); 7) compute α, β, τ, η, δ, ν from Eqs. (13, 14, 15, 16, 17, 18); 8) compute F, G, T'', A', B' from Eqs. (7, 8, 9, 10); 9) solve Eq. (4) for γ; 10) compute Q and R from Eqs. (11) and (12); 11) compute $\boldsymbol{\varepsilon}$ from Eq. (5); 12) compute $\mathbf{r}_2 = \mathbf{r}_1 + \boldsymbol{\epsilon}$, $r_2 = (\mathbf{r}_2\cdot\mathbf{r}_2)^{1/2}$; 13 compute P_2 and S_2 [equations after (3)]; 14) compute ρ and σ from Eqs. (23, 19, 20); 15) compute $\boldsymbol{\lambda}$ from Eq. (6). Since γ will usually be small, $1 - \cos\gamma$ should be replaced by $2\sin^2(\gamma/2)$ for computational purposes.

Numerical Example

We assume the two particles to be in coplanar circular orbits with units chosen such that $a_1^3 = \mu$. The initial conditions are

$$x_{10} = 1,\ y_{10} = 0,\ \dot{x}_{10} = 0,\ \dot{y}_{10} = 1$$

$$\epsilon_{x0} = 0.001,\ \epsilon_{y0} = 0,\ \lambda_{x0} = 0,\ \lambda_{y0} = -0.0004996253122$$

If we let $t = \pi/4$, solve for $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{v}_1$, $\mathbf{v}_2$ at time t and compute $\boldsymbol{\varepsilon}$ and $\boldsymbol{\lambda}$ from the differences $\mathbf{r}_2 - \mathbf{r}_1$ and $\mathbf{v}_2 - \mathbf{v}_1$, we obtain

$$\epsilon_x = 0.0015394491,\ \epsilon_y = -0.0001262154$$

$$\lambda_x = 0.0011853623,\ \lambda_y = 0.0004778069$$

Using the formulas developed in this paper we obtain

$$\epsilon_x = 0.001539449086,\ \epsilon_y = -\ 0.0001262154558$$

$$\lambda_x = 0.001185362260,\ \lambda_y = 0.0004778069038$$

A ten-digit calculator was used for these calculations.

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Fatigue Life Estimation of Fluttering Panels

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It is now generally appreciated that the development of nonlinear flutter analyses of plates and shells permits the estimation of fatigue life.[1] Such analyses provide not only the dynamic pressure, Mach number, etc., at which flutter begins but also the flutter frequency and stress levels as one penetrates beyond the flutter boundary and into the flutter regime itself. Knowing the stress levels and frequency in the flutter regime, one may use a conventional fatigue curve (stress vs number of cycles to fatigue) to estimate fatigue life for the fluttering panel. It is the purpose of the present Note to provide some examples of fatigue life estimation and discuss their implications for design.

For simplicity, we shall consider an isotropic, rectangular flat plate simply supported on all edges. The principal parameters are (see Ref. 1 for notation)

$$\bar{\sigma}_x = (\sigma_x/E)(a/h)^2\,(1 - \nu^2),\ \text{stress}$$

$$K^2 \equiv \omega^2\rho_m h a^4/D,\ \text{frequency (squared)}$$

$$\lambda^* \equiv 2qa^3/D,\ \text{dynamic pressure}$$

M, Mach number; $\mu = \rho a/\rho_m h$, mass ratio; a/b, length/width. The stress $\bar{\sigma}_x$, and frequency K are determined from the nonlinear flutter analysis when the other parameters, λ^*, M, μ, a/b, are given. Knowing $\bar{\sigma}_x$, K one can determine the dimensional stress, σ_x, and frequency, $f = \omega/2\pi$ for a given panel. From a conventional fatigue curve, given σ_x one may determine the number of cycles to fatigue failure N. The fatigue life is then given by $T = N/f$. For simplicity, we approximate the fatigue curve by an algebraic formula

$$N = (2 \times 10^5/\sigma_x)^6 \tag{1}$$

Equation (1) is a reasonable approximation for aluminum. Furthermore, we shall use the maximum tensile stress in Eq. (1) to determine N. The stress in the panel varies with both position and time. A more complicated rule for determining N could be used where available fatigue data warrant. In particular, the temporal variation of σ_x takes the form of a sinusoidal variation about a mean tensile stress level. In a more precise fatigue life estimate, this mean tensile stress would be taken into account.

In our examples, we shall concentrate on a length/width ratio of two and, for the most part, high supersonic Mach number. Because of the latter, the three parameters λ^*, μ, M may be reduced to two, $\lambda^*/(M^2 - 1)^{1/2}$ and μ/M. In

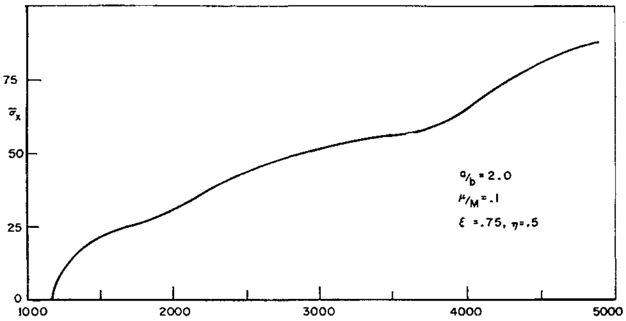

Fig. 1 Stress vs dynamic pressure.

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